Pergamon

*Int. J. Solids Structures* Vol. 35, No 30, pp. 3917–3933, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(97)00260–6

# ASYMPTOTIC DESCRIPTION OF THE LOGARITHMIC SINGULAR STRESS FIELD AND ITS APPLICATION

Y. Y. YANG*
Institute for Reliability and Failure Analysis, University of Karlsruhe, P.O. Box 3640, D-76021 Karlsruhe, Germany

(*Received* 9 *July* 1996; *in revised form* 26 *August* 1997)

**Abstract**—In a joint of two dissimilar materials under mechanical or thermal loading the stresses at the intersection of the edges and the interface are very high or singular for elastic materials behaviour. For most joint geometries and material combinations there is a type of $r^{-\omega}$ singularity. However, for some joint geometries and material combinations there is a type of $\ln(r)$ singularity. In this paper, the type of $\ln(r)$ stress singularity for a two dissimilar materials joint under thermal loading is treated by the Mellin transform method. Emphasis is placed on the asymptotical description of the stress distribution near the singular point. The angular functions used to describe the stresses are given in a general form, which are different to those of the same joint under edge tractions. For a quarter-planes joint angular functions are given in an explicit form. Finally, examples are presented to show when the asymptotical solution for the type of $\ln(r)$ stress singularity can be used to describe stresses near the singular point in the practical relevant joints. For a finite joint the unknown factor, $K$, used to describe the stress is given as well. © 1998 Elsevier Science Ltd. All rights reserved.

## NOTATIONS

| | |
|---|---|
| $\Phi$ | stress function |
| $T$ | temperature change |
| $q$ | see eqn (1) |
| $E$ | Young's modulus |
| $\nu$ | Poisson ratio |
| $\alpha$ | thermal expansion coefficient and Dundurs parameter |
| $\beta$ | Dundurs parameter |
| $r$ | polar coordinates |
| $\bar{r}$ | $r/L$, $L$ is a characteristic length of a joint |
| $\theta$ | polar coordinates |
| $R_0$ | radius with temperature change in a semi-infinite joint |
| $\bar{R}_0$ | $R_0/L$ |
| $T_0$ | temperature change in a semi-infinite joint |
| $s$ | Mellin transform parameter |
| $\sigma_{ij}$ | stress tensor |
| $\sigma$ | normal stress component |
| $\tau$ | shear stress component |
| $u$ | displacement component in the $r$ direction |
| $v$ | displacement component in the $\theta$ direction |
| $\|X\|$ | see eqn (14) |
| $g_{ij}$ | see eqn (15), eqn (14) |
| $p_{ij}$ | see eqn (16), eqn (14) |
| $f_{ij}$ | see eqn (20), eqn (25) |
| $t_{ij}$ | see eqn (21), eqn (26) |
| $l_{ij}$ | see eqn (22) |
| $\omega$ | stress exponent, $\omega = s+2$ |

## 1. INTRODUCTION

In many technical applications, dissimilar materials have to be joined together. Due to the difference in elastic properties and thermal expansion coefficients of the components joined, high stresses occur at the intersection of the edges and the interface or at the interface corner under mechanical or thermal loading. Changes in temperature cause thermal stresses

* Fax: 00 49 7247 822 347. E-mail: yang@imf.fzk.de.

as a result of the different thermal expansion coefficients. This is important especially if joining is done at high temperatures. In the sense of linear elasticity, stress singularity exists for most material combinations at the intersection of the edges and the interface of the joint (denoted as a singular point).

In the last ten years, many studies were published about analyses of the stress singularity in a joint under mechanical or thermal loading. Some of them deal with the dependence of the order of the singularity on the wedge angles and on the material constants for a joint with free edges (see e.g., Williams, 1952; Bogy, 1968; Hein and Erdogan, 1971; Dempsey and Sinclair, 1981; van Vroonhoven, 1992; Vasilopoulos, 1988; Theocaris, 1974), for a joint with edge tractions (see Bogy, 1971), for a joint with an interface corner (see Bogy and Wang, 1971) and for a joint with free–fix or fix–fix edges (see e.g. Williams, 1952; Dempsey and Sinclair, 1981). Others cover the stress distribution near the singular point in a joint with free edges (Munz and Yang, 1992; Knesl *et al.*, 1991, Blanchard and Ghoniem, 1989, 1990; Suga *et al.*, 1989), in a joint with an interface corner (see Yang and Munz, 1995) and in a joint with edge tractions (see Yang and Munz, 1997). All of them apply to the type of $r^{-\omega}$ singularity, i.e., the stresses near the singular point can be described by

$$\sigma_{ij}(r,\theta) = \sum_{n=1}^{N} \frac{K_n}{(r/L)^{\omega_n}} h_{ijn}(\theta) + \sigma_0 h_{ij0}(\theta),$$

where $r$ and $\theta$ are the coordinates (see Fig. 1), $L$ is a characteristic length of the joint, $\omega_n$ is real, and $\omega_n > 0$.

There is also another type of stress singularity. Bogy and Dempsey (see Bogy, 1970; Bogy, 1971; Bogy and Wang, 1971; Dempsey and Sinclair, 1979, 1981) described the conditions of a two dissimilar materials joint with the type of $\ln(r)$ or $r^{-\omega}\ln(r)$ singularity. Bogy (see Bogy, 1970) studied the type of $\ln(r)$ singularity in a quarter-planes joint under

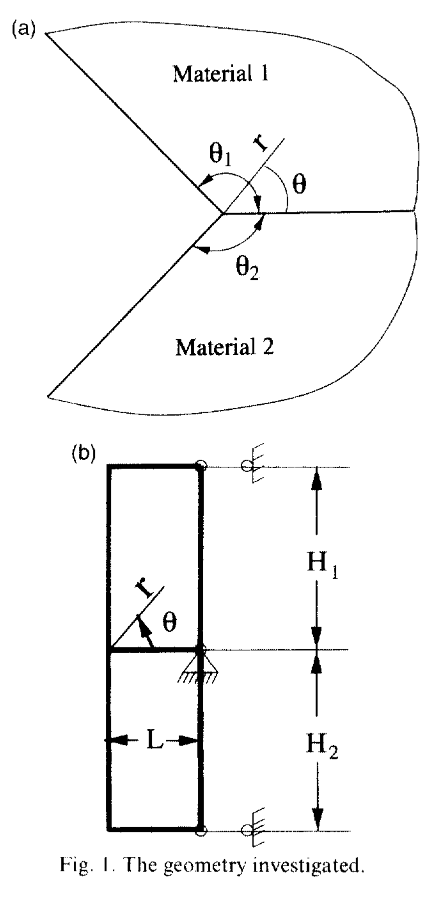


Fig. 1. The geometry investigated.

edge tractions. Dempsey (see Dempsey, 1995) examined special cases with an $r^{-\omega}\ln(r)$ singularity. However, there has been no study, so far, of the asymptotical description of the stress distribution for a type of $\ln(r)$ singularity in a two dissimilar materials joint under thermal loading, i.e., the angular functions and the factor $K$ used to describe the stress distribution near the singular point are unknown. In this paper, the type of $\ln(r)$ stress singularity is treated by the Mellin transform method for a joint with free edges under thermal loading. Emphasis is placed on the asymptotical description of the stress distribution near the singular point. For a finite joint angular functions and the unknown factor, $K$, used to describe the stress are given. For a quarter-planes joint, angular functions are given in an explicit form. The angular functions for a joint under thermal loading are different to those shown by Bogy (see Bogy, 1970) for a joint under edge tractions.

Although the type of $\ln(r)$ singularity is unlikely to occur in practice, the meaning of this paper is to complete the solution of the singularity problems. The most important application of the solution for the type of $\ln(r)$ singularity is that it can be used to describe the singular stress field well for material combinations with very small stress exponent $\omega$ in the type of $r^{-\omega}$ singularity, which will be seen in the given examples. For joints under thermal loading this is useful, because for material combinations with very small stress exponent $\omega$ ($\omega \to 0$) the solution from the type of $r^{-\omega}$ singularity is not stable.

## 2. SOLUTION IN THE MELLIN TRANSFORM DOMAIN

When thermal loading is taken into account and body forces are disregarded, the stress function, $\Phi$, should satisfy the equation

$$\nabla^4\Phi + q\nabla^2 T = 0 \tag{1}$$

with

$$q = \begin{cases} \alpha E & \text{for plane stress} \\ \dfrac{\alpha E}{(1-\nu)} & \text{for plane strain} \end{cases},$$

where $T$ is the temperature change, $E$ is Young's modulus, $\nu$ is the Poisson ratio, $\alpha$ is the thermal expansion coefficient. To find an analytical solution of eqn (1), the Mellin transform method is used, because in Mellin domain eqn (1) is replaced by an ordinary differential equation. In the Mellin transform method a semi-infinite space is considered. Therefore, at first a semi-infinite joint with the temperature change

$$T = \begin{cases} T_0 & \text{for } r \leqslant R_0 \\ 0 & \text{for } r > R_0 \end{cases}$$

is considered (see Fig. 1).

The Mellin transform of a function, $\Phi(\bar{r}, \theta)$, is defined as

$$\hat{\Phi}(s, \theta) = \int_0^{\infty} \Phi(\bar{r}, \theta)\bar{r}^{(s-1)}\,\mathrm{d}\bar{r} \tag{2}$$

where $s$ is a parameter of the Melling transform, and $\bar{r}$ is dimensionless ($\bar{r} = r/L$, $L$ is a characteristic length of the joint). The parameters should be chosen so that the integration in eqn (2) is valid. The property of the Mellin transform is

$$\int_0^\infty \bar{r}^p \frac{\partial^q \Phi(\bar{r},\theta)}{\partial \bar{r}^q} \bar{r}^{s-1}\,\mathrm{d}\bar{r} = (-1)^q \frac{\Gamma(s+p)}{\Gamma(s+p-q)} \hat{\Phi}(s+p-q,\theta) \tag{3}$$

where $\Gamma(x)$ is the $\Gamma$-function.

The Mellin transform of eqn (1) then reads

$$\left[s^2+\frac{\partial^2}{\partial\theta^2}\right]\left[(s+2)^2+\frac{\partial^2}{\partial\theta^2}\right]\hat{\Phi}(s,\theta)+\left[(s+2)^2+\frac{\partial^2}{\partial\theta^2}\right]\hat{T}(s+2)=0 \tag{4}$$

with

$$\hat{T}(s+2)=\int_0^\infty qT\bar{r}^{(s+1)}\,\mathrm{d}\bar{r}=\frac{qT_0}{s+2}\bar{R}_0^{(s+2)} \tag{5}$$

where $\bar{R}_0 = R_0/L$.

If $s$ is considered as a parameter, eqn (4) is an ordinary differential equation of the variable $\theta$. Its solution is

$$\hat{\Phi}_k(s,\theta)=A_k e^{is\theta}+\bar{A}_k e^{-is\theta}+B_k e^{i(s+2)\theta}+\bar{B}_k e^{-i(s+2)\theta}-\frac{\hat{T}_k(s+2)}{s^2} \tag{6}$$

where $A_k$, $B_k$ ($\bar{A}_k, \bar{B}_k$ is the conjugate complex number of $A_k$, $B_k$) are unknown and $k = 1$, 2 for materials 1 and 2. In order to determine the unknown $A_k$, $B_k$ the boundary conditions expressed by the Mellin transform are used for a joint with free edge. They are

$$\begin{aligned}\theta=\theta_1:&\quad \hat{\tau}_{1r\theta}+i\hat{\sigma}_{1\theta\theta}=0\\ \theta=\theta_2:&\quad \hat{\tau}_{2r\theta}+i\hat{\sigma}_{2\theta\theta}=0\end{aligned} \tag{7}$$

at the interface

$$\begin{aligned}\theta=0:&\quad \hat{\tau}_{1r\theta}+i\hat{\sigma}_{1\theta\theta}=\hat{\tau}_{2r\theta}+i\hat{\sigma}_{2\theta\theta}\\ \theta=0:&\quad \hat{u}_1+i\hat{v}_1=\hat{u}_2+i\hat{v}_2\end{aligned} \tag{8}$$

where $u$ and $v$ are the displacement in the direction $r$ and $\theta$. The physical meaning of eqn (7) is that at the free edges $\theta = \theta_1$ and $\theta = \theta_2$ the normal stress $\sigma_\theta$ and the shear stress $\tau_{r\theta}$ are zero. The physical meaning of eqn (8) is that at the interface $\theta = 0$ the normal stress $\sigma_\theta$, the shear stress $\tau_{r\theta}$ and the displacements $u$ and $v$ should be continuous in materials 1 and 2. To use the boundary conditions the stresses and the displacements have to be transformed in the Mellin domain.

The Mellin transform of the stress components is

$$\begin{aligned}\hat{\sigma}_{rr}(s,\theta)&=\left(\frac{\partial^2}{\partial\theta^2}-s\right)\hat{\Phi}(s,\theta)\\ \hat{\sigma}_{\theta\theta}(s,\theta)&=(s+1)s\hat{\Phi}(s,\theta)\\ \hat{\tau}_{r\theta}(s,\theta)&=(s+1)\frac{\partial\hat{\Phi}(s,\theta)}{\partial\theta}.\end{aligned} \tag{9}$$

The complex form of the Mellin transform of stresses and displacements is

$$\hat{\tau}_{r\theta}(s,\theta)+i\hat{\sigma}_{\theta\theta}(s,\theta)=(s+1)\left(\frac{\partial}{\partial\theta}+is\right)\hat{\Phi}(s,\theta) \tag{10}$$

and

$$2G\{\hat{u}(s+1,\theta)+i\hat{v}(s+1,\theta)\} = -\frac{\kappa\hat{T}(s+2)}{4(s+1)} + \left(s-i\frac{\partial}{\partial\theta}\right)\left\{1+\frac{\kappa\left(\frac{\partial}{\partial\theta}-is\right)\left(\frac{\partial}{\partial\theta}-i(s+2)\right)}{4(s+1)(s+2)}\right\}\hat{\Phi}(s,\theta) \tag{11}$$

with

$$\kappa=\begin{cases}\dfrac{4}{(1+\nu)} & \text{for plane stress}\\ 4(1-\nu) & \text{for plane strain}\end{cases}.$$

To obtain a real description of the stresses and displacements, the coefficients, $A_1$, $B_1$ and $A_2$, $B_2$, are separated as

$$A_1=C_1+iD_1,\quad A_2=C_2+iD_2$$
$$B_1=F_1+iH_1,\quad B_2=F_2+iH_2. \tag{12}$$

Then the Mellin transform of the stress function is real in

$$\hat{\Phi}_k(s,\theta)=2\{C_k\cos(s\theta)-D_k\sin(s\theta)+F_k\cos((s+2)\theta)-H_k\sin((s+2)\theta)\}-\frac{\hat{T}_k(s+2)}{s^2}. \tag{13}$$

After substitution of eqn (13) into eqn (10) and eqn (11) then into eqn (7) and eqn (8), the coefficients, $C_k$, $D_k$, $F_k$ and $H_k$ can be determined (for details, see Yang and Munz, 1994).

From Yang and Munz (1994) we have the following relations for stresses in the Mellin domain:

$$\hat{\sigma}_{ij}(s,\theta)=\frac{p_{ij}(s,\theta)g_{ij}(s)}{\|X\|(s+2)} \tag{14}$$

where

$$g_{rr}(s)=\frac{T_0\bar{R}_0^{s+2}}{2s},$$
$$g_{\theta\theta}(s)=\frac{T_0\bar{R}_0^{s+2}}{2s}(s+1),$$
$$g_{r\theta}(s)=-\frac{T_0\bar{R}_0^{s+2}}{s}(s+1), \tag{15}$$

$$p_{ij}(s,\theta)=A_{1ij}(s)\cos(s\theta)+A_{2ij}(s)\sin(s\theta)+A_{3ij}(s)\cos((s+2)\theta)+A_{4ij}(s)\sin((s+2)\theta)+A_{5ij}(s) \tag{16}$$

and $\|X\|$ is the determinant of a matrix. The definitions of $\|X\|$ and $A_{lij}(s)$ with $ij = rr$, $\theta\theta$, $r\theta$ and $1 = 1, 2, 3, 4, 5$, see Yang and Munz (1994) in eqns (30)–(32).

For example,

$$
\begin{aligned}
A_{1rr} &= -C^*2(s+1), \\
A_{2rr} &= D^*2(s+1), \\
A_{3rr} &= -F^*2(s^2+5s+4), \\
A_{4rr} &= H^*2(s^2+5s+4), \\
A_{5rr} &= 2q\|X\|,
\end{aligned}
$$

where $C^*$, $D^*$, $F^*$, $H^*$ are contained in eqns (26) and (28) of the paper by Yang and Munz (1994). They are a function of the material properties $E_1$, $E_2$, $\nu_1$, $\nu_2$, the angles $\theta_1$, $\theta_2$, and the parameter $s$. All quantities, $\|X\|$ and $A_{lij}$, can be obtained analytically.

## 3. SOLUTION IN A POLAR COORDINATE SYSTEM

Our aim is to calculate the stresses in a polar coordinates system, i.e. $\sigma_{ij}(s, \theta)$, which is the reversal transform of $\hat{\sigma}_{ij}(s, \theta)$. For the calculation of the reversal of $\hat{\sigma}_{ij}(s, \theta)$ we need the poles of $\hat{\sigma}_{ij}(s, \theta)$, which are defined as follows: if $\lim_{s \to s_k} \hat{\sigma}_{ij}(s, \theta) \to \infty$, $s_k$ is the pole of $\hat{\sigma}_{ij}(s, \theta)$. From eqn (14) it can be seen that the possible poles of $\hat{\sigma}_{ij}(s, \theta)$ are the solutions of $\|X\| = 0$ and $s = -2$.

From the definition of the reversion of the Mellin transform, the stresses in a polar coordinate system can be calculated by

$$\sigma_{ij}(\bar{r}, \theta) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \hat{\sigma}_{ij}(s, \theta) \bar{r}^{-(s+2)} \, \mathrm{d}s \tag{17}$$

where $\gamma$ must be chosen so that the integration in eqn (17) exists. According to the residual principle stresses in the polar coordinate system can be obtained from

$$
\begin{aligned}
\sigma_{ij}(r, \theta) &= \sum_{s_k < \gamma} res\{\hat{\sigma}_{ij}(s_k, \theta)\bar{r}^{-(s_k+2)}\} \\
&= \sum_{s_k < \gamma} \frac{1}{(m-1)!} \lim_{s \to s_k} \frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}} \{(s-s_k)^m \hat{\sigma}_{ij}(s, \theta)\bar{r}^{-(s+2)}\} \\
&= \sum_{s_k < \gamma} \frac{1}{(m-1)!} \lim_{s \to s_k} \frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}} \left\{(s-s_k)^m \frac{p_{ij}(s, \theta) g_{ij}(s)}{\|X\|(s+2)} \bar{r}^{-(s+2)}\right\}
\end{aligned}
\tag{18}
$$

where $s_k$ is the $m$-th order pole of $\hat{\sigma}_{ij}(s, \theta)$. In this paper, we consider only the case with $m = 2$ and $s_k = -2$, i.e. $s = -2$ is the second order pole of $\hat{\sigma}_{ij}(s, \theta)$, which corresponds to the type of $\ln(r)$ singularity. From eqn (18) and the equations in Yang and Munz (1994) we know that is the case with $\|X\||_{s=-2}$, $[(\mathrm{d}\|X\|)/\mathrm{d}s]|_{s=-2} = 0$, $[(\mathrm{d}^2\|X\|)/\mathrm{d}s^2]|_{s=-2} \neq 0$ and $p_{ij}(s, \theta)|_{s=-2} = 0$, $[(\partial p_{ij}(s, \theta))/\partial s]|_{s=-2} \neq 0$, because for $s = -2$ there is always $g_{ij}(s) \neq 0$. For the case with $m > 2$, the calculation for the reversal of stress $\hat{\sigma}_{ij}(s, \theta)$ is more complicated, but the procedure is similar to that given here for $m = 2$.

For $m = 2$ and $s = -2$, the stresses in polar coordinates can be calculated from

$$\sigma_{ij}(r,\theta) = \lim_{s\to-2}\frac{\mathrm{d}}{\mathrm{d}s}\left\{(s+2)^2\bar{r}^{-(s+2)}\frac{p_{ij}(s,\theta)g_{ij}(s)}{\|X\|(s+2)}\right\}$$

$$= -g_{ij}(s)|_{s=-2}\ln(\bar{r})\lim_{s\to-2}\frac{p_{ij}(s,\theta)(s+2)}{\|X\|}$$

$$+g_{ij}(s)|_{s=-2}\lim_{s\to-2}\frac{\frac{\partial p_{ij}(s,\theta)}{\partial s}(s+2)\|X\|+\|X\|p_{ij}(s,\theta)-\frac{\mathrm{d}\|X\|}{\mathrm{d}s}(s+2)p_{ij}(s,\theta)}{\|X\|^2}$$

$$+\left.\frac{\mathrm{d}g_{ij}(s)}{\mathrm{d}s}\right|_{s=-2}\lim_{s\to-2}\frac{p_{ij}(s,\theta)(s+2)}{\|X\|}. \tag{19}$$

In the following, definitions

$$f_{ij}(\theta) = \frac{1}{2}\lim_{s\to-2}\frac{p_{ij}(s,\theta)(s+2)}{\|X\|}$$

$$= \left.\frac{\dfrac{\partial p_{ij}(s,\theta)}{\partial s}}{\dfrac{\mathrm{d}^2\|X\|}{\mathrm{d}s^2}}\right|_{s=-2}, \tag{20}$$

$$t_{ij}(\theta) = \lim_{s\to-2}\frac{\frac{\partial p_{ij}(s,\theta)}{\partial s}(s+2)\|X\|+\|X\|p_{ij}(s,\theta)-\frac{\mathrm{d}\|X\|}{\mathrm{d}s}(s+2)p_{ij}(s,\theta)}{\|X\|^2}$$

$$= \left.\frac{\dfrac{\mathrm{d}^2\|X\|}{\mathrm{d}s^2}\dfrac{\partial^2 p_{ij}(s,\theta)}{\partial s^2}-\dfrac{2}{3}\dfrac{\mathrm{d}^3\|X\|}{\mathrm{d}s^3}\dfrac{\partial p_{ij}(s,\theta)}{\partial s}}{\left(\dfrac{\mathrm{d}^2\|X\|}{\mathrm{d}s^2}\right)^2}\right|_{s=-2}, \tag{21}$$

and

$$l_{ij} = g_{ij}(s)|_{s=-2} \tag{22}$$

will be used, where $l_{rr} = -(T_0/4)$, $l_{\theta\theta} = (T_0/4)$ and $l_{r\theta} = -(T_0/2)$. From eqn (15) we have

$$\left.\frac{\mathrm{d}g_{ij}(s)}{\mathrm{d}s}\right|_{s=-2} = \frac{1}{2}l_{ij}(K+I_{ij}) \tag{23}$$

with $K = 2\ln(\bar{R}_0)$ and $I_{rr} = 1$, $I_{\theta\theta} = -1$, $I_{r\theta} = -1$. The stresses near the singular point in polar coordinates finally can be rewritten as

$$\sigma_{ij}(r,\theta) = l_{ij}\{-2\ln(r/L)f_{ij}(\theta)+t_{ij}(\theta)+(K+I_{ij})f_{ij}(\theta)\}. \tag{24}$$

In eqn (24), the quantity $\sigma_{ij}(r,\theta)$ has the same unit as the Young's modulus, i.e., MPa or GPa, $K$ is dimensionless, $f_{ij}(\theta)$ and $t_{ij}(\theta)$ have the unit of $E_k * \alpha_k$, and $l_{ij}$ has the unit of temperature. The functions, $f_{ij}(\theta)$ and $t_{ij}(\theta)$, can be calculated analytically from eqns (20) and (21), because $p_{ij}(s,\theta)$ and $\|X\|$ are known. They depend on the material parameters $(E_1, E_2, \nu_1, \nu_2, \alpha_1, \alpha_2)$ of the components joined and the angles $\theta_1$ and $\theta_2$. $l_{ij}$ is proportional to thermal loading. For a semi-infinite joint, the factor $K$ is known.

Generally, the functions $f_{ijk}(\theta)$ and $t_{ijk}(\theta)$ ($k = 1$, 2 for materials 1 and 2), have this form:

$$f_{ijk}(\theta) = F_{1ijk}\cos(2\theta)+F_{2ijk}\sin(2\theta)+F_{3ijk}+F_{4ijk}\theta \tag{25}$$

$$t_{ijk}(\theta) = T_{1ijk}\cos(2\theta)+T_{2ijk}\sin(2\theta)+T_{3ijk}+T_{4ijk}\cos(2\theta)\theta+T_{5ijk}\sin(2\theta)\theta+T_{6ijk}\theta. \tag{26}$$

The coefficients, $F_{lijk}$ and $T_{nijk}$ with $ij = rr, \theta\theta, r\theta$; $l = 1, 2, 3, 4$ and $n = 1, 2, 3, 4, 5, 6$, can be calculated analytically by substituting eqn (16) into eqns (20) and (21). For an arbitrary geometry (i.e., arbitrary $\theta_1, \theta_2$), the relations between the coefficients $F_{lijk}$, $T_{nijk}$, the material properties ($E_1, E_2, \nu_1, \nu_2, \alpha_1, \alpha_2$), and the geometry $\theta_1$, $\theta_2$ have a very long form. However, for a quarter-planes joint (i.e., $\theta_1 = -\theta_2 = 90^\circ$), they are simple, which are given in an explicit form as follows by using the REDUCE-code. For the coefficients $F_{lijk}$ there is:

$$F_{4ijk} = 0 \tag{27}$$

$$F_{Lij1} = F_{Lij2} \quad \text{for } L = 1,2,3 \tag{28}$$

$$F_{1rr1} = -F_{3rr1} = F_{1\theta\theta1} = F_{3\theta\theta1} = -2F_{2r\theta1} = \frac{16Q\alpha}{Z} \tag{29}$$

$$F_{Lr\theta k} = 0, \quad \text{for } L = 1,3 \tag{30}$$

$$F_{2IJk} = 0, \quad \text{for } IJ = rr, \theta\theta. \tag{31}$$

For the coefficients $T_{nijk}$ there is:

$$T_{1rr1} = -\frac{16Q}{Z^2}(Z(\alpha+2\beta-\theta_1^2)+32(\alpha-\beta)((3\alpha-2\beta)(\alpha-2\beta)+\theta_1^2)) \tag{32}$$

$$T_{1rr2} = -\frac{16Q}{Z^2}(Z(\alpha+2\beta+\theta_1^2)+32(\alpha-\beta)((3\alpha-2\beta)(\alpha-2\beta)+\theta_1^2)) \tag{33}$$

$$T_{2rr1} = -\frac{16Q}{Z}(\alpha-1)\theta_1 \tag{34}$$

$$T_{2rr2} = \frac{16Q}{Z}(\alpha+1)\theta_1 \tag{35}$$

$$T_{4IJk} = 0 \quad \text{for } IJ = rr, \theta\theta \tag{36}$$

$$T_{5rr1} = T_{5rr2} = \frac{32Q}{Z}\alpha \tag{37}$$

$$T_{3rr1} = \frac{16Q}{Z^2}(Z((2\alpha+1)\theta_1^2-(\alpha-2\beta))+32(\alpha-\beta)((3\alpha-2\beta)(\alpha-2\beta)+\theta_1^2)) \tag{38}$$

$$T_{3rr2} = \frac{16Q}{Z^2}(Z((2\alpha-1)\theta_1^2-(\alpha-2\beta))+32(\alpha-\beta)((3\alpha-2\beta)(\alpha-2\beta)+\theta_1^2)) \tag{39}$$

$$T_{6rr1} = -\frac{32Q}{Z}(\alpha+1)\theta_1 \tag{40}$$

$$T_{6rr2} = \frac{32Q}{Z}(\alpha-1)\theta_1 \tag{41}$$

$$T_{1\theta\theta 1} = \frac{16Q}{Z^2}(Z(\alpha-2\beta+\theta_1^2)-32(\alpha-\beta)((3\alpha-2\beta)(\alpha-2\beta)+\theta_1^2)) \tag{42}$$

$$T_{1\theta\theta 2} = \frac{16Q}{Z^2}(Z(\alpha-2\beta-\theta_1^2)-32(\alpha-\beta)((3\alpha-2\beta)(\alpha-2\beta)+\theta_1^2)) \tag{43}$$

$$T_{N\theta\theta k} = T_{Nrrk} \quad \text{for } N = 2, 5 \tag{44}$$

$$T_{N\theta\theta k} = -T_{Nrrk} \quad \text{for } N = 3, 6 \tag{45}$$

$$T_{1r\theta 1} = -T_{6rr2}/4 \tag{46}$$

$$T_{1r\theta 2} = T_{2rr2}/2 \tag{47}$$

$$T_{2r\theta k} = -T_{1\theta\theta k}/2 \tag{48}$$

$$T_{4r\theta k} = T_{5rrk}/2 \tag{49}$$

$$T_{3r\theta 1} = T_{2rr2}/2 \tag{50}$$

$$T_{3r\theta 2} = T_{2rr1}/2 \tag{51}$$

$$T_{Nr\theta k} = 0 \quad \text{for } N = 5, 6 \tag{52}$$

with

$$Z = 8(-2\alpha\beta\theta_1^2+\theta_1^2+5\alpha^2-10\alpha\beta+4\beta^2) \tag{53}$$

$$Q = \frac{\Delta\alpha}{\dfrac{1}{E_1'}+\dfrac{1}{E_2'}}$$

$$\Delta\alpha = \begin{cases} \alpha_1-\alpha_2 & \text{for plane stress} \\ \alpha_1(1+\nu_1)-\alpha_2(1+\nu_2) & \text{for plane strain} \end{cases},$$

$$E_k' = \begin{cases} E_k & \text{for plane stress} \\ \dfrac{E_k}{1-\nu_k^2} & \text{for plane strain} \end{cases}, \tag{54}$$

where $\alpha$, $\beta$ are the Dundurs parameters defined as

$$\alpha = \frac{\kappa_2-g\kappa_1}{\kappa_2+g\kappa_1}$$

$$\beta = \frac{(\kappa_2-2)-g(\kappa_1-2)}{\kappa_2+g\kappa_1} \tag{55}$$

with $g = (G_2/G_1)$.

## 4. RESULTS AND DISCUSSIONS

In this Section, four examples will be presented to show the application of the asymptotic solution of the logarithmic singular stress field in a quarter-planes joint. A comparison of the stresses from FEM and the analytical description as in eqn (24) for the type of $\ln(r)$ singularity is given at first to show the good agreement of them. Then the use of the analytical description for the type of $\ln(r)$ singularity is presented to evaluate the singular stresses analytically, without using any numerical method, for joints with very small $\omega$ in the type of $r^{-\omega}$ singularity.

Theoretically, if the stress exponent $\omega$ ($\omega = 2+s$, and $s$ is the solution of $\|X\| = 0$) is not zero, the stresses near the singular point in a joint under thermal loading should be described by

$$\sigma_{ij}(r,\theta) = \sum_{n=1}^{N} \frac{K_n}{(r/L)^{\omega_n}} h_{ijn}(\theta) + \sigma_{ij0}(\theta), \tag{56}$$

where $\sigma_{ij0}(\theta)$ is the regular stress term and can be determined analytically (see Yang and Munz, 1992; Munz *et al.*, 1993; Yang and Munz, 1994), $h_{ijn}(\theta)$ are the angular functions for non-logarithmic stress singularity and can also be determined analytically (see Munz and Yang, 1994). The distance, $r$, is divided by $L$ so that the factors, $K_n$, have the unit of stress. For an arbitrary geometry and an arbitrary materials combination, there may be more than one singular term. Equation (56) includes all singular terms in the sum, i.e., in eqn (56), $\omega_n > 0$. For a quarter-planes joint there is $N = 1$.

If the value of $\omega$ is very small, however, the use of eqn (56) is questionable. The equation $\|X\| = 0$ is a transcendental equation. To solve it a numerical method has to be used. For very very small $\omega$ ($\omega < 10^{-6}$), the solution is sensitive to the explicit form of $\|X\|$. In fact, for the same $\|X\|$ explicit form using different numerical method to solve $\|X\| = 0$ the solution is different as well. This means that it is difficult to determine the accurate value of $\omega$ for a materials combination with a very very small $\omega$. Therefore, using eqn (56) to describe stresses near the singular point in a joint under thermal loading is difficult. In fact, from the sense of the physical meaning, the exact value of $\omega$ is not important for the joint with very very small $\omega$. On the other hand, for a two dissimilar materials joint under thermal loading, due to the effect of the regular term $\sigma_{ij0}(\theta)$ the case of $\omega \to 0$ does not mean that the stress singularity disappears. However, this cannot be seen from eqn (56) obviously.

From mathematics it is known that if $\omega \to 0$ there is also $[(\mathrm{d}\|X\|)/\mathrm{d}s]|_{s=s_k} \to 0$ ($s_k = \omega - 2$). Now the question arises whether eqn (24) can be used to calculate the stresses near the singular point in a joint with very small $\omega$ under thermal loading. If so, how small can $\omega$ be?

Although eqn (24) is deduced from the case of a semi-infinite joint with the temperature change

$$T = \begin{cases} T_0 & \text{for } r \leqslant R_0 \\ 0 & \text{for } r > R_0, \end{cases}$$

it can be used in a finite joint with a homogeneous temperature change to calculate the stresses near the singular point (it should be noted that this solution is used only in the area near the singular point). This means that the angular functions are the same for a finite joint as for a semi-infinite joint; only, for a finite joint, the quantity $R_0$ is unknown and therefore, the factor $K$ in eqn (24) is unknown. It has to be determined from the stresses calculated by FEM.

The method to determine the factor $K$ for a finite joint will be simply presented as follows. In eqn (24) the quantities $f_{ij}(\theta)$, $t_{ij}(\theta)$, $l_{ij}$ and $I_{ij}$ can be calculated analytically, if the stresses are known from the FEM, we can define one quantity $\Pi$:

$$\Pi_{ij} = \sum_{l=1}^{M} \{\sigma_{ij}^{\mathrm{FEM}}(r_l, \theta_l) - l_{ij}\{-2\ln(r_l/L) f_{ij}(\theta_l) + t_{ij}(\theta_l) + (K + I_{ij}) f_{ij}(\theta_l)\}\} \tag{57}$$

where there is $ij = xx$, $yy$, $xy$, or $rr$, $\theta\theta$, $r\theta$, $M$ is the number of the used points for the determination of the $K$ factor. In principle, any stress component at any point $(r_l, \theta_l)$ near the singular point can be used. In general, we use the points along a line, i.e. $\theta_l$ is a constant. Following the least square method the factor $K$ can be determined from

$$\frac{\partial \Pi_{ij}}{\partial K} = 0. \tag{58}$$

For the calculation of the stresses from FEM a standard element with eight nodes is used. The mesh needs not to be very fine.

In eqns (20) and (21), the functions $f_{ij}$ and $t_{ij}$ are given in polar coordinates. By using the transform between polar coordinates and Cartesian coordinates,

$$\begin{aligned} f_x &= f_r \cos^2(\theta) + f_\theta \sin^2(\theta) - 2 f_{r\theta} \sin(\theta)\cos(\theta) \\ f_y &= f_r \sin^2(\theta) + f_\theta \cos^2(\theta) + 2 f_{r\theta} \sin(\theta)\cos(\theta) \\ f_{xy} &= (f_r - f_\theta)\cos(\theta)\sin(\theta) + f_{r\theta}(\cos^2(\theta) - \sin^2(\theta)) \end{aligned} \tag{59}$$

we can obtain the corresponding quantities in Cartesian coordinates. For a quarter-planes joint, after transforming $f_{ij} l_{ij}$ from polar coordinates to Cartesian coordinates, there is

$$\begin{aligned} f_x l_x &= 0 \\ f_y l_y &= \frac{T_0}{2} F_{1\theta\theta 1} = \frac{T_0}{2} F_{1\theta\theta 2} \\ f_{xy} l_{xy} &= 0. \end{aligned} \tag{60}$$

From eqns (24) and (60) we know that, in Cartesian coordinates the stress components $\sigma_x$ and $\tau_{xy}$ are independent of $r$ (the distance from the singular point) and the factor $K$. Therefore, $\sigma_x$ and $\tau_{xy}$ are not singular. Only the stress component $\sigma_y$ is singular.

The results given in the following are for plane strain. The geometry is $H_1/L = H_2/L = 2$ (see Fig. 1). For the FEM-calculation the ABAQUS-code was used with eight nodes standard element. The mesh near the singular point is fine. The smallest length in the element is about $10^{-5}L$.

*Example* 1

The materials data for Example 1 are

$$E_1 = 100\,\mathrm{GPa}, \quad \nu_1 = \tfrac{1}{3}, \quad \alpha_1 = 2.5 * 10^{-6}/\mathrm{K},$$

$$E_2 = 54\,\mathrm{GPa}, \quad \nu_2 = 0.2, \quad \alpha_2 = 8.5 * 10^{-6}/\mathrm{K}.$$

For this joint, the stress exponent is

$$\omega = 0,$$

and

$$\left.\frac{\mathrm{d}\|X\|}{\mathrm{d}s}\right|_{s=-2} = 0,$$

$$\left.\frac{\mathrm{d}^2\|X\|}{\mathrm{d}s^2}\right|_{s=-2} \neq 0.$$

This is the case of $\alpha = 2\beta$ for a quarter-planes joint.

The factor $K$ in eqn (24) was determined by using eqn (58) and FEM. For this example there is

$$K = 0.4424$$

which is the averaged value of those calculated from eqn (58) with $\theta_l = -90^\circ, 90^\circ, -45^\circ$, $45^\circ$ and $0^\circ$. Using the $K$-factor as determined we can calculate the stresses analytically at an arbitrary point with eqn (24). Comparison of the stresses obtained from FEM and with eqn (24) along $\theta = 0$ is shown in Fig. 2. The quantities used to calculate the stresses with eqn (24) in Cartesian coordinates system are

| $\theta^\circ$ | $\sigma_{ij}$ | $f_{ij}$, GPa/K | $t_{ij}$, GPa/K |
|---|---|---|---|
| 0 | $\sigma_x$ | 0 | $-2.1863 * 10^{-4}$ |
| 0 | $\sigma_y$ | $-1.4899 * 10^{-4}$ | $6.8670 * 10^{-4}$ |
| 0 | $\tau_{xy}$ | 0 | $-3.5106 * 10^{-4}$. |

The results show that the stresses calculated by FEM and with eqn (24) are in excellent agreement in the range near the singular point. So, we can say that eqn (24) can describe very well the stresses near the singular point analytically for the type of $\ln(r)$ singularity.

For the same joint, but with remote mechanical loading, i.e., at the upper and lower surfaces of the joint there is a homogeneous stress (e.g., $\sigma_y = 1$ MPA); the stress distribution in the joint was calculated by FEM. The stress distribution along $\theta = 0$ is plotted in Fig. 3. It can be seen that there is no stress singularity in this joint under remote mechanical loading.

Therefore, we can say that $\|X\||_{s=-2} = 0$, $[(\mathrm{d}\|X\|)/\mathrm{d}s]|_{s=-2} = 0$, and $[(\mathrm{d}^2\|X\|)/\mathrm{d}s^2]|_{s=-2} \neq 0$ are not all conditions for the type of $\ln(r)$ singularity. It also depends on the loading.

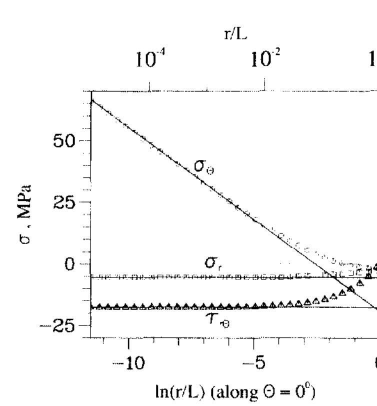


Fig. 2. Comparison of the stresses calculated by FEM (symbols) and with eqn (24) (lines) along $\theta = 0$ for Example 1 under thermal loading.

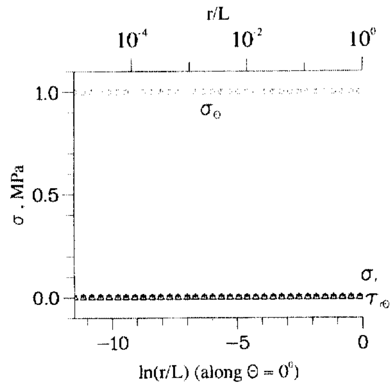


Fig. 3. Stress distribution along $\theta = 0$ for Example 1 under mechanical loading.

*Example* 2

The materials data for Example 2 are

$$E_1 = 100\,\text{GPa}, \quad \nu_1 = 0.333, \quad \alpha_1 = 2.5 * 10^{-6}/\text{K},$$
$$E_2 = 54\,\text{GPa}, \quad \nu_2 = 0.2, \quad \alpha_2 = 8.5 * 10^{-6}/\text{K}.$$

For this joint, the stress exponent is

$$\omega = 6.0222 * 10^{-5}$$

and

$$\left.\frac{\mathrm{d}|X|}{\mathrm{d}s}\right|_{s=s_k} = 1.3880 * 10^{-4}.$$

Although the conditions for the type of $\ln(r)$ singularity are not satisfied exactly, eqn (24) is used to calculate the stresses near the singular point. The obtained $K$ factor in eqn (24) is

$$K = 0.3856.$$

The quantities used to calculate the stresses with eqn (24) are

| $\theta^\circ$ | $\sigma_{ij}$ | $f_{ij}$, GPa/K | $t_{ij}$, GPa/K |
|---|---|---|---|
| $-45^\circ$ | $\sigma_r$ | $7.44588 * 10^{-5}$ | $7.79347 * 10^{-6}$ |
| $-45^\circ$ | $\sigma_\theta$ | $-7.44588 * 10^{-5}$ | $6.94198 * 10^{-4}$ |
| $-45^\circ$ | $\tau_{r\theta}$ | $-3.72294 * 10^{-5}$ | $2.38395 * 10^{-4}$. |

A comparison of the stresses obtained from FEM and with eqn (24) along $\theta = -45^\circ$ is

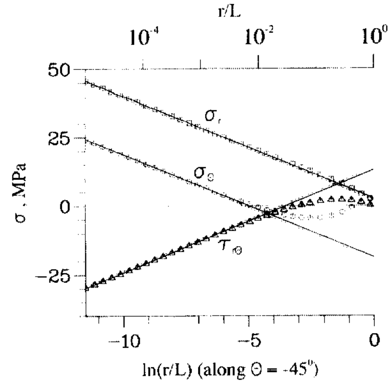


Fig. 4. Comparison of the stresses calculated by FEM (symbols) and with eqn (24) (lines) along $\theta = -45^\circ$ for Example 2.

shown in Fig. 4. The results show that, for this materials combination (i.e., the stress exponent $\omega$ is approximate $6 * 10^{-5}$), eqn (24) can be used as well to describe very well the stresses near the singular point, and that, although the stress exponent $\omega$ is very small the stress singularity is obvious.

Because the absolute value of the term $K f_{ij} l_{ij}$ is relatively lower than that of the term $\ln(r) f_{ij} l_{ij}$ for a very small distance $r$, we know from eqn (24) that the stresses calculated from eqn (24) are not sensitive to the accuracy of the $K$ factor determined. To see the effect of the value of $K$ in eqn (24) on stresses in Fig. 4 the stresses calculated from eqn (24) with $K = 0$ are also plotted as dashed lines. In Fig. 4 the solid lines indicate the results with $K = 0.3856$. It can be seen that the effect of the value of $K$ on stress is not strong. When the absolute value of $K$ is lower, the effect of $K$ on stresses is smaller.

In the following two examples will be given to show for which materials combination eqn (24) with $K = 0$ can be used to calculate the stress distribution near the singular point.

*Example* 3

As Example 3 a real material combination is chosen, which is a $Si_3N_4/W$ joint. The materials data are

$$E_1 = 314\,\text{GPa}, \quad \nu_1 = 0.28, \quad \alpha_1 = 2.7 * 10^{-6}/\text{K},$$

$$E_2 = 411\,\text{GPa}, \quad \nu_2 = 0.28, \quad \alpha_2 = 4.5 * 10^{-6}/\text{K}.$$

For this joint, the stress exponent is

$$\omega = 0.0056.$$

The stresses were calculated by FEM and from eqn (24) with $K = 0$. The quantities used to calculate the stresses with eqn (24) are

| $\theta^\circ$ | $\sigma_{ij}$ | $f_{ij}$, GPa/K | $l_{ij}$, GPa/K |
|---|---|---|---|
| $-45^\circ$ | $\sigma_r$ | $-4.7979 * 10^{-5}$ | $7.9641 * 10^{-4}$ |
| $-45^\circ$ | $\sigma_\theta$ | $4.7979 * 10^{-5}$ | $3.3018 * 10^{-4}$ |
| $-45^\circ$ | $\tau_{r\theta}$ | $-2.3990 * 10^{-5}$ | $6.5716 * 10^{-5}$. |

A comparison of the stresses obtained from FEM and from eqn (24) with $K = 0$ along

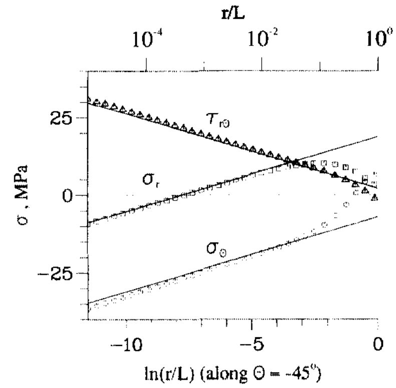


Fig. 5. Comparison of the stresses calculated by FEM (symbols) and with eqn (24) (lines) along $\theta = -45°$ for Example 3.

$\theta = -45°$ is shown in Fig. 5. The results show that, for this materials combination (i.e., the stress exponent $\omega$ is approximate $6 * 10^{-3}$) eqn (24) with $K = 0$ can be used as well to describe well the stresses near the singular point (with error $< 6\%$ for $r/L < 10^{-2}$).

*Example* 4

The materials data for Example 4 are

$$E_1 = 100\,\text{GPa}, \quad \nu_1 = 0.3, \quad \alpha_1 = 2.5 * 10^{-6}/\text{K},$$

$$E_2 = 50\,\text{GPa}, \quad \nu_2 = 0.2, \quad \alpha_2 = 8.5 * 10^{-6}/\text{K}.$$

For this joint, the stress exponent is

$$\omega = 9.6826 * 10^{-3}.$$

The stresses were calculated by FEM and from eqn (24) with $K = 0$. The quantities used to calculate the stresses with eqn (24) are

| $\theta°$ | $\sigma_{ij}$ | $f_{ij}$, GPa/K | $t_{ij}$, GPa/K |
|---|---|---|---|
| $-45°$ | $\sigma_r$ | $7.4797 * 10^{-5}$ | $-2.1864 * 10^{-5}$ |
| $-45°$ | $\sigma_\theta$ | $-7.4797 * 10^{-5}$ | $6.8028 * 10^{-4}$ |
| $-45°$ | $\tau_{r\theta}$ | $-3.7399 * 10^{-5}$ | $2.3596 * 10^{-4}$. |

A comparison of the stresses obtained from FEM and from eqn (24) with $K = 0$ along $\theta = -45°$ is shown in Fig. 6. The results show that for this materials combination (i.e., the

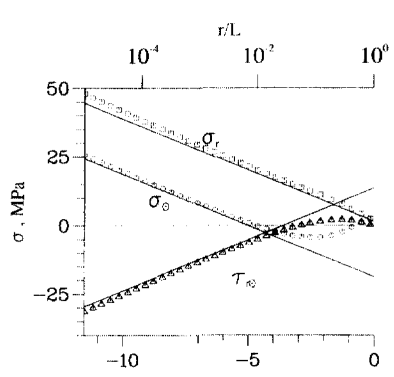


Fig. 6. Comparison of the stresses calculated by FEM (symbols) and with eqn (24) (lines) along $\theta = -45°$ for Example 4.

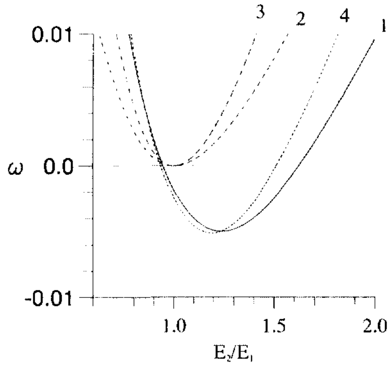


Fig. 7. Stress exponent $\omega$ for some practical relevant material combinations: curve 1 is for $\nu_1 = 0.2$, $\nu_2 = 0.3$; curve 2 is for $\nu_1 = 0.2$, $\nu_2 = 0.2$; curve 3 is for $\nu_1 = 0.3$, $\nu_2 = 0.3$; curve 4 is for $\nu_1 = 0.25$, $\nu_2 = 0.35$.

stress exponent $\omega$ is approximate $10^{-2}$), eqn (24) with $K = 0$ can be used to describe the stresses near the singular point with error smaller than 10% for $r/L < 10^{-2}$.

From Figs 4–6 it can be seen that the equations for the type of $\ln(r)$ singularity with $K = 0$ can be used to calculate stresses near the singular point if the stress exponent $\omega$ in the type of $r^{-\omega}$ is very small. It should be noted that FEM is not needed to calculate the stresses near the singular point in a joint with very small $\omega$ by using eqn (24) with $K = 0$.

For practical relevant materials, i.e. $0.2 < \nu_1 < 0.4$ and $0.2 < \nu_2 < 0.4$, material combinations with $0.6 < E = E_2/E_1 < 2$ have usually very small stress exponent $\omega$, see Fig. 7. Therefore, the application of the asymptotic description of the logarithmic singular stress field is of interest. Numerical calculations show also that if $\omega$ is negative, but absolute value very small, eqn (24) with $K = 0$ can be used to calculate stresses near the singular point as well.

## 5. CONCLUSIONS

In this paper, the type of $\ln(r)$ singularity in a two dissimilar materials joint with a free edge under thermal loading has been studied by the Mellin transform method.

If $s = -2$ (i.e. $\omega = 0$) is the second order pole ($m = 2$) of $\hat{\sigma}_{ij}(s,\theta) = [(p_{ij}(s,\theta)g_{ij}(s))/(\|X\|(s+2))]$ in the Mellin domain, there is the type of $\ln(r)$ singularity. This is the case with $\|X\||_{s=-2} = 0$, $[(\mathrm{d}\|X\|)/\mathrm{d}s]|_{s=-2} = 0$, $[(\mathrm{d}^2\|X\|)/\mathrm{d}s^2]|_{s=-2} \neq 0$ and $p_{ij}(s,\theta)|_{s=-2} = 0$, $[(\partial p_{ij}(s,\theta))/\partial s]|_{s=-2} \neq 0$.

For the case with $m = 2$ and $s = -2$, in an arbitrary finite joint under thermal loading stresses near the singular point can be calculated from

$$\sigma_{ij}(r,\theta) = l_{ij}\{-2\ln(r/L)f_{ij}(\theta) + t_{ij}(\theta) + (K + I_{ij})f_{ij}(\theta)\}. \tag{61}$$

For a quarter-planes joint ($\theta_1 = -\theta_2 = 90^\circ$), the angular functions $f_{ij}(\theta)$ and $t_{ij}(\theta)$ in polar coordinates were given in an explicit form. The $K$ factor can be determined by using the stresses calculated by FEM and the least squares method. For the type of $\ln(r)$ singularity the effect of the accuracy of the determined $K$ on the stress near the singular point is small.

For a quarter-planes joint in Cartesian coordinates, the stress components $\sigma_x$ and $\tau_{xy}$ are independent of $r$ (the distance from the singular point) and the factor $K$. Therefore, $\sigma_x$ and $\tau_{xy}$ are not singular. Only the stress component $\sigma_y$ is singular.

The case of $\omega \to 0$ does not mean that the stress singularity disappears for a joint with free edge under thermal loading. For joints with very small $\omega$ ($\omega < 10^{-2}$) in the type of $r^{-\omega}$ singularity, stresses near the singular point can be described also well by the equations for the type of $\ln(r)$ stress singularity. Especially, the equations for the type of $\ln(r)$ stress singularity with $K = 0$ can be applied to calculate stresses near the singular point for joints with $H_1/L > 1$ and $H_2/L > 1$. This means that the stresses near the singular point can be calculated without using any FEM. The error is less than 10% for $\omega < 0.01$ and for $\omega < 0.005$ the error is less than 5% with $r/L < 10^{-2}$. If the corresponding value of $K$ in eqn (24) is used, the error is smaller.

*Acknowledgement*—The author would like to thank Prof. D. Munz for his useful comments and discussions.

## REFERENCES

Blanchard, J. P. and Ghoniem, N. M. (1989) An eigenfunction approach to singular thermal stresses in bonded strips. *Journal of Thermal Stresses* **12**, 501–527.

Blanchard, J. P. and Ghoniem, N. M. (1990) Analysis of singular stress fields in duplex fusion components. *Journal of Nuclear Materials* **172**, 54–70.

Bogy, D. B. (1968) Edge-bonded dissimilar orthogonal elastic wedges under normal and shear loading. *Transactions of the ASME, Journal of Applied Mechanics* **35**, 460–466.

Bogy, D. B. (1970) On the problem of edge-bonded elastic quarter-planes loaded at the boundary. *International Journal of Solids and Structures* **6**, 1287–1313.

Bogy, D. B. (1971) Two edge-bonded elastic wedges of different materials and wedge angles under surface tractions. *Transactions of the ASME, Journal of Applied Mechanics* **38**, 377–386.

Bogy, D. B. and Wang, K. C. (1971) Stress singularities at interface corners in bonded dissimilar isotropic elastic materials. *Journal of Solids and Structures* **7**, 993–1005.

Dempsey, J. P. and Sinclair, G. B. (1979) On the stress singularities in the plane elasticity of the composite wedge. *Journal of Elasticity* **9**, 373–391.

Dempsey, J. P. and Sinclair, G. B. (1981) On the singular behavior at the vertex of a bi-material wedge. *Journal of Elasticity* **11**, 317–327.

Dempsey, J. P. (1995) Power-logarithmic stress singularities at bi-material corners and interface cracks. *J. Adhesion Sci. Technol.* **9**, 253–265.

Hein, V. L. and Erdogan, F. (1971) Stress singularities in a two-material wedge. *International Journal of Fracture and Mechanics* **7**, 317–330.

Knesl, Z., Sramek, A., Kad'ourek, J. and Kroupa, F. (1991) Stress concentration at the edge of coatings on tensile specimens. *Acta Techn., CSAV* **36**, 574–593.

Munz, D., Fett, T. and Yang, Y. Y. (1993) The regular stress term in bonded dissimilar materials after a change in temperature. *Engineering Fracture Mechanics* **44**, 185–194.

Munz, D. and Wang, Y. Y. (1994) Stresses near the free edge of the interface in ceramic to metal joints. *Journal of the European Ceramic Society* **13**, 453–460.

Suga, T., Mizuno, K. and Miyazawa, K. (1989) Thermal stresses in ceramic-to-metal joints. *MRS Int. Mtg on Adv. Mats.* **8**, 137–142.

Theorcaris, P. S. (1974) The order of singularity at a multi-wedge corner in a composite plate. *International Journal of Engineering Science* **12**, 107–120.

van Vroonhoven, J. C. W. (1992) Stress singularities in bi-material wedges with adhesion and delamination. *Fatigue Fract. Engng Mater. Struct.* **15**, 159–171.

Vasilopoulos, D. (1988) On the determination of higher order terms of singular elastic stress fields near corners. *Numer. Math.* **53**, 51–95.

Williams, M. L. (1952) Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *Transactions of the ASME, Journal of Applied Mechanics* **74**, 526–528.

Yang, Y. Y. and Munz, D. (1992) A method of determination of the regular stress term for an arbitrary joint geometry under thermal loading. KfK—Report, No. 5089, Karlsruhe.

Yang, Y. Y. and Munz, D. (1994) Determination of the regular stress term in a dissimilar materials joint under thermal loading by the Mellin transform. *Journal of Thermal Stresses* **17**, 321–336.

Yang, Y. Y. and Munz, D. (1995) Stress intensity factor and stress distribution in a joint with an interface corner under thermal and mechanical loading. *Journal of Computers and Structures* **57**, 467–476.

Yang, Y. Y. and Munz, D. (1997) Stress singularities in a dissimilar materials joint with edge tractions under mechanical and thermal loading. *International Journal of Solids and Structures* **34**, pp. 1199–1216.